"'QI' Maths and geometric means"

Ken McKelvie : UoL Maths Club 31 March 2012

0. Summary

1. Maths Thinking in a Nutshell

A schematic representation of your maths thinking processes through levels of increasing refinement might be:

"Unsorted (chaotic) thinking" Ↄ "Systematic plausible reasoning" Ↄ "Proof"

(Notation: "Ↄ" can be read as "includes")

"Proof" characterises mathematics.

"Plausible reasoning" is any informed "guesswork" which you believe is likely to be true. Possibly it is short of proof but, at the next step, it may lead on to proof.

It features in training through elements of problem solving in Management Science and in Medicine. In these disciplines absolute proof is a doubtful luxury

This type of reasoning appears to underlie at least two strands of mathematical activity. At a basic level it seems to describe methodologies which can be employed as a "double-check" as to whether, or not, a mathematical conclusion is likely to be valid. At a higher it provides the signposts for the exploration of "unknown territory" leading to conjectures which in turn become candidates for formal proof.

2. Aims: Exploration and SWAP

This presentation has two main aims.

The first aim is to help you explore some things about yourselves. These include, for example, why you might get stuck when attempting either to solve a problem or to understand more deeply a new (to you) idea in mathematics.

The title of the presentation was nearly "Blank minds, Blind spots and Pitfalls" in the senses of Blank Minds: when you have no idea as to how to proceed, Blind Spots: when you have half a right idea but something is missing, and Pitfalls: when you do have an idea (maybe a longstanding one) but in practice it turns out to be a wrong one.

The second aim is to explore a simple strategy, summarised as "**SWAP** approaches", which might provide pointers hopefully helping you to overcome being stuck. Here **SWAP** represents the leading letters of "**S**ymbols, **W**ords **A**nd **P**ictures", all of which, in the broadest sense, are the most basic of mathematical tools with which you work.

In pursuit of both of these aims we are exploring skills in the area of systematic plausible reasoning.

3. About 'QI'

Both for some light relief and as an aid to our exploration, 'QI' type questions will be posed, and - as in the tv programme - there will be neither shame nor blame associated with not knowing the answers !

The success of the tv programme is built on human curiosity.
See http://www.qi.com/about/philosophy.php
This website notes that pure curiosity, completely standard in children under seven and found in great artists, scientists and explorers, is, for some reason, quickly suppressed, sublimated or shrunken in most people as they grow. How about yourselves?

A subsidiary aim of this presentation is to re-sharpen some of the acuity of your curiosity that you may be in danger of losing – or even perhaps already have lost.

(For more on 'QI' see the British Comedy Guide site
'QI' About the show http://www.comedy.co.uk/guide/tv/qi/
'QI' Overview http://www.comedy.co.uk/guide/tv/qi/about/)

'QI' q1. *Amongst comedy shows televised in Britain during 2011 – 2012 season , which had the biggest demand for audience tickets, and, at the same time, the second biggest demand amongst all TV shows after* Top Gear? *Have a guess.*

4. Some symbols

'QI' q2. *What is the widely accepted evolution of the "+" symbol?*
'QI' q3. *What was the nationality of the "inventor" of the "=" symbol c 1557?*

See http://jeff560.tripod.com/operation.html
See http://jeff560.tripod.com/relation.html

5. <u>"Means means Means"</u>

There many and various types of mean: for example,

- Arithmetic mean
- Generalized mean also known as **power mean** or **Hölder mean**
- **Geometric mean**
- Harmonic mean
- Heronian mean
- Quadratic mean also known as the root-mean-square

For details, see http://en.wikipedia.org/wiki/Mean

Here we will focus on "arithmetic mean values" and "geometric mean values".

'QI' q4. *What is the core of the meaning of "arithmetic"?*
See http://en.wikipedia.org/wiki/Arithmetic ....

'QI' q5. *What is the core of the meaning of "geometric"?*
See http://en.wikipedia.org/wiki/Geometry ....

The array of particular types of mean value listed above begs the question: What is meant by a mean value?

The English language is somewhat paradoxical in that there are several (many?) different words having the same meaning, albeit with different nuances. On the other hand the English Language often uses single words which having several different meanings.
Ironically "mean" is one such word.

'QI' q6. *The Microsoft Word Thesaurus list for "mean" includes as synonyms one equivalent noun, five distinct equivalent verbs and three distinct equivalent adjectives. Hazard a guess as to what these might be.*

.... , ... , ... , ... , ...

.... , ... , ... , ... , resource.
The final synonym "resource", for "mean", shown above does not appear in the Thesaurus list. It is commonly used in the plural eg "ways and means".

In the context of the slice of mathematics that we are addressing here, "mean value" is used in the sense of "intermediate value" or, informally, "in-between value". Did you **<u>really</u>** know this?

'QI' q7. *As an aid to the memory, there was a popular music-hall song in the 1920's called "Ain't we got fun?"*
It includes the line "In the mean-time, in-......... time, ain't we got fun?"
Guess what's omitted here .
Hear it at http://www.youtube.com/watch?v=w1qSFaYxVXk

We are eventually going to explore, for the simplest case of two values *a,b*, how we define their arithmetic mean value and, with $a>0$ and $b>0$, how we define their geometric mean value. We next present a text-book algebraic derivation of a simple inequality between their arithmetic mean value and their geometric mean value, and then look for ways in which this inequality might be derived by alternative SWAP approaches.

Below are some general wise words of caution. Please read through details later.

6. Words of Caution (i) Beware of "acting blindly" involuntarily

When faced with particular circumstances it is wise to recognise that evolution has left us with a number of instinctive involuntary reactions over which might perhaps have little prior control.

A possibly serious situation is an individual's reaction to an escalation of frustration to a sense of "breaking point". It can be regarded as akin being faced with potential conflict resulting in the momentary response of "fight or flight". Adrenaline involuntarily pumps to your fists or to your legs without in the instant your being in control of it. This can lead to an unintended outcome – which is one reason why youngsters carrying knives is not a good idea - "I didn't mean to kill him !"
In mathematical education it is the stage at which many learners perhaps decide not to fight their teacher but opt for flight and "give up" with a declaration that they "can't do maths".

Another example, of relevance here, is the facility of the mind to compensate for minor discrepancies and, in a sense, make them invisible. The brain receives images via the eyes in a pixcel form. Minor Infrastructural deterioration in the vision system will result in gaps which the brain fills in from experience and the mental image formed will appear continuous and complete. There is then, seemingly, no consequential functional deterioration to be detected. If individuals are aware of vision deficiencies and have blind spots, then they have to be careful because "they don't know what they can't see". Be aware, some people also suffer from mental blind-spots of which they might not be aware.

For example, when proof-reading hard-copy, it is advisable to use some technique other than just re-reading alone. This could involve using a guide ruler and a moving finger. The rationale for this is because what the mind sees on the hard-copy during re-reading alone is what it expects to see rather than what is actually there.

Finally a third example of an involuntary response comes with the feeling that someone is either following you or just watching you.

7. <u>Words of Caution – (ii) Beware of "acting blindly" as a result of unquestioned early learning</u>

Much early "pre-school education" comes in two forms: (i) partly through the satisfaction of innocent curiosity but also (ii) through the formal learning process of "receive, repeat by rote, regurgitate". This latter process sometimes results in the learner accumulating a disconnected set of terms and ideas which requires greater effort to retain and recall than if relevant basic relationships were recognised from the outset.

Illustrations:
'QI' q8. Words. *What are the plural and the collective of "experience"?*

.... , ...

'QI' q9. Words. *What are the plural and the collective of "sheep"?*

.... , ...

'QI' q10. *In the nursery rhyme, what four words follow "Baa baa black sheep"?*

....

'QI' q11. *How many front teeth does a sheep have on its upper jaw?* ....

http://www.childrensuniversity.manchester.ac.uk/interactives/science/teethandeating/discovermore/gallery/sheep.asp

'QI' q12. *In the nursery rhyme, what six words follow "Hickory, dickory, dock"?*

....

'QI' q13. *What is the usually accepted version in the UK of the origination of the words "Hickory, dickory, dock"?*

....

http://en.wikipedia.org/wiki/Yan_Tan_Tethera

'QI' q14. *In the same context as the answer to q10, what is the origination of the meaning of "score" having the meaning 20?*

....

8. <u>Words of Caution – (iii) Beware the "Two Ronnies" fork handles scenario</u>

In this situation, the same or similar sounding words can have very different meanings depending on the context of use. Communication between two people or two groups of people can fail if this is not recognised. Always seek an alternative interpretation and, if there is one, double-check as soon as practicable which is the intended one.

9. Words of Caution - (iv) Beware of forgetting to remember

In brief, your recollection of anything that you encounter is dependent on its rehearsal (ie recycling, equivalently revisiting and turning over) in your mind immediately following the encounter. This rehearsal may be by design or it may be involuntary such as following a trauma or other crisis.

You can usefully conjecture that you have two memory stores, one being your short-term “memory” and the other your long-term “memory”. All contents of the stores start life in the short-term memory. Those that are rehearsed transfer to your long-term memory, the remainder decay rapidly and are gone typically within three days.

‘QI’ q15. This is an illustrative exercise that you can try “anytime”:
For each of the three preceding weeks attempt to recall the total numbers of hours that you spent variously -
(i) Asleep (ii) On Academic Work (iii) On Other Activity.
(NB: For each week the sum of the totals of your numbers of hours should be 168).

Easy or Hard?

10. Arithmetic and Geometric Means: Definitions

The following is a typical introduction which we will subsequently shortly unpick and then explore issues arising.

Rather than attempt to start from the most general definitions of the various types of mean values of sets of say n values, we will for clarity here restrict ourselves initially to the simplest case of a set of two values.

We will denote these values of these as a,b (or, indeed, sometimes as A,B or as x,y, etc).

Arithmetic mean:

‘QI’ q16. *What is the core of the meaning of “arithmetic”?*
http://en.wikipedia.org/wiki/Arithmetic

Alongside a,b we introduce a further two values l,m say such that the value m is midway between a and b ie $m – a = b – m$ and we let this common difference be l . We can think in terms of progress from a to $m = a + l$ and then from $m = a + l$ to $b = m + l$. ie $m = a +2l$,

eg for $a = 4$, $b = 36$, then $m = 20$ and $2l = 32$, ie $l = 16$.

The sequence *a, m, b,* here 4, 20, 36, is known as an arithmetic sequence - or arithmetic progression.

The sequence could be extended: 4, 20, 36, 52, 68,.. and so on, but here we are restricting our attention initially to the values 4, 20, 36)

In terms of *a,b* it can be shown that (i) $m = \frac{a+b}{2}$ and (ii) $l = \frac{b-a}{2}$

The value $m = \frac{a+b}{2}$ is defined to be the **arithmetic mean value** of ***a,b***
[A complication, which in practice I confess that I have never encountered so perhaps it can safely be ignored is as follows.

Any number of "arithmetic mean values", say three, can be fitted between
a = 4, *b* = 36.

These are values are to be equi-spaced between *a* = 4, *b* = 36 computed as follows.
Number of 'gaps' between *a* = 4 and *b* = 36 and is to be 3 + 1 = 4, each of length (36 – 4)/4 = 8. The "three arithmetic mean values" will be 12, 20, 28.]

Geometric mean:

'QI' q17. *What is the core of the meaning of "geometric"?*
http://en.wikipedia.org/wiki/Geometry

We set the conditions a > 0, b > 0.

We introduce a further two values r,*g* say such that the ratio *g/a* is equal to the ratio *b/g* and let these common ratios be *r.*
We can think in terms of progress from *a* to *g* = *ar* and then from *g* = *ar* to *b* = *gr* *ie to b* = ar^2.

eg for *a* = 4, *b* = 36, then *g* = 12 and *r* = 3 .

The sequence *a, g, b,* here 4, 12, 36, is known as geometric sequence - or geometric progression.

The sequence could be extended: 4, 12, 36, 108, 324, ... and so on, but here we are restricting our attention initially to the values 4, 12, 36)

In terms of *a,b* it can be shown that (i) $g = \sqrt{ab}$ and (ii) $r = \sqrt{b/a}$
The value $g = \sqrt{ab}$ is defined to be the **geometric mean value** of ***a,b.***

[A complication, as for arithmetic means, is that any number of geometric mean values can similarly be fitted between *a* = 4, *b* = 36. But not here !]

11. Arithmetic and Geometric Means: Inequality

In the preceding example, we note that $m = 20 > g = 12$
so we have "arithmetic mean > geometric mean ".

We will now show that that this inequality holds in general in respect of our two values *a,b* when $a \neq b$.

We know that $(\sqrt{a} - \sqrt{b})^2 > 0$

So $a - 2\sqrt{ab} + b > 0$

ie $a + b > 2\sqrt{ab}$

ie $\frac{a+b}{2} > \sqrt{ab}$

so: when $a \neq b$, arithmetic mean > geometric mean

When a = b, the ">" sign in our initial inequality becomes an "=" sign, which carries through the subsequent argument and leads to the conclusion that
when a = b, arithmetic mean = geometric mean.

12. Generalisations of 10,11 for Future Reference
- but not followed-through here.

For *n* values, $a_1, a_2, a_3, \ldots, a_n$, their arithmetic mean value *m* is defined to be

$$m = \frac{a_1 + a_2 + \cdots . + a_n}{n}$$

For *n* values, $a_1 > 0, a_2 > 0, a_3 > 0, \ldots, a_n > 0$, their geometric mean value *g* is defined to be

$$g = \sqrt[n]{a_1 a_2 a_3 \ldots . a_n}$$

Further it can be shown that :
Arithmetic mean $\geq$ geometric mean.

13. Arithmetic and Geometric Means - by Pictures

Next, we have an example of first-principles reasoning which relates the elements of the arithmetic and geometric means inequality for two values A, B in pictorial form The steps are developed through the appropriate interpretation of the following sequence of four related figures.

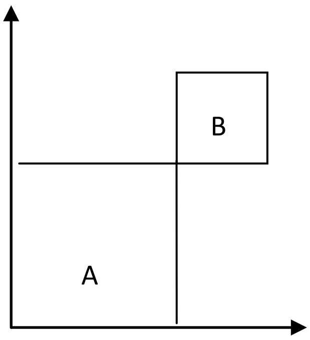

Fig 1

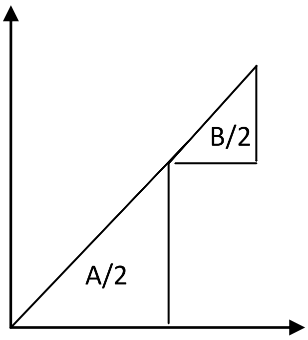

Fig 2

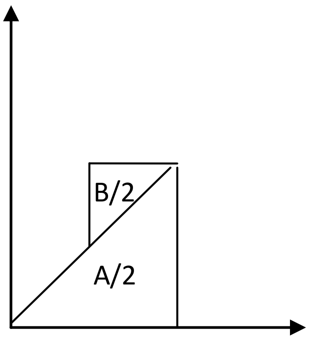

Fig 3

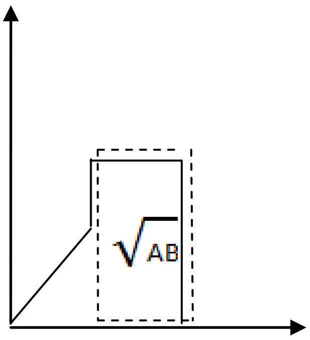

Fig 4

Figure 1, with A ≥ B, shows a larger square, with area A and sides length √A, and a smaller square, with area B, and sides length √B. For ease of visualisation, the leading diagonals of the two squares are collinear.

Figure 2 shows the two triangles having areas A/2 and B/2 formed by slicing the given squares along their leading diagonals and then disposing of the upper halves.

Figure 3 is obtained by visualising the two triangles hinged at their common corner and then rotating the smaller to its new position as shown. Refer to the resulting composite figure as an axe-head. The area of the axe-head is (A/2 + B/2) = (A + B)/2, ie the arithmetic mean of A and B.

Figure 4 shows the axe-head decomposed as a dashed rectangle and its residual triangle. The lengths of the sides of the rectangle are √A and √B respectively. Its area is thus √(AB), ie the geometric mean of A and B. The area of the residual triangle is always positive except when A=B.

Hence for A and B we have arithmetic mean $\geq$ geometric mean, with equality holding only when A=B.

For a comprehensive selection of further examples see:

http://jwilson.coe.uga.edu/emt725/AMGM/AMGM.html

14. Tail Piece : Promoting Enlightenment!

Sometimes Maths Outreach activities, such as Maths Club, may be described as aiming to “Engage, Enlighten, Enrich” the learners.

For lots on Maths, see the Cambridge NRICH site http://nrich.maths.org/public/

Finally, for distant future reference, an inequality relating four means derived at a level of technical complexity to which at present you might only aspire, see http://www.artofproblemsolving.com/Wiki/index.php/Root-Mean_Square-Arithmetic_Mean-Geometric_Mean-Harmonic_mean_Inequality

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